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Sobolev Estimates for Fractional and Singular Radon Transforms

SCIPIO CUCCAGNA*

Department of Mathematics, Columbia University, New York, New York 10027

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We prove Sobolev inequalities for singular and fractional Radon transforms which are defined as in a paper of Phong and Stein under certain hypothesis on the corresponding Lagrangian ($(N^*C)'$) which does not necessarily have to be a canonical graph. In the proof we use oscillatory integrals, the Cotlar–Stein almost orthogonality theorem, a sort of Littlewood–Paley decomposition for a certain operator, some basic facts about Fourier integral operators and pseudodifferential operators. The main ideas come from papers by Phong and Stein (*Acta Math.* **157** (1986), 99–157) and Sogge and Stein (*J. Analyse Math.* **54** (1990), 165–188).

INTRODUCTION

This paper is an outgrowth of reference [7]. As in [7] we consider a manifold X of dimension $n+d$, in [7] $d=1$ and here more generally $0<d\leqslant(n^2+n)/2$. We consider $C\subset X\times X$ a submanifold of codimension d containing the diagonal. We suppose that the two projections π_L resp. π_R of C on the left resp. right factor of $X\times X$ have maximal rank and the projections $d\pi_L$ resp. $d\pi_R$ of the conormal bundle N^*C of C on the the left resp. right factor of $T^*X\times T^*X$ have rank everywhere not smaller than $n+2d+k$, where $1\leqslant k\leqslant n$. In case $k=n$ then $(N^*C)'$ is a canonical graph. Observe that for $d=\operatorname{codim} C>n^2$ our last hypothesis cannot hold as can be seen from expression (1.5) below (in theory we could consider case $(n^2+n)/2<d\leqslant n^2$ but here we want to state results valid generically for the case $S(t,x,y)=S(t,x-y)$ for the function in (1.4) below). We consider an open neighbourhood C' in C of the diagonal and on it an admissible coordinate system ι (the following notions come from p. 106 [7]), that is, a covering of C' by open sets (C_j) with a C^∞ function ι_j defined in each (C_j) satisfying

(1) $\iota_j|_{C_j\cap\Delta}=0$ (Δ is the diagonal in $X\times X$);

(2) $\iota_j(P,Q)\in\mathbb{R}^n$ is for each fixed P a coordinate system (y) in C'_P where $\{P\}\times C'_P=(\{P\}\times C_P)\cap C'$ and $\{P\}\times C_P=\pi_L^{-1}(\{P\})$

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E-mail address: cuccagna@math.columbia.edu.

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(3) $\pi_L(C_j)$ is included for each j in some coordinate patch with coordinates (x).

Then, given $K \in C_0^\infty(C')$, an admissible coordinate system $\imath$, integers $M, N \geqslant 0$ and a fixed number μ with $0 < \mu \leqslant n = \dim C_P$, we consider (see (1.2) in [7])

$$\|K\|_{M,N}^{\imath,j} = \sup \sum_{|\alpha| \leqslant N,\ |\beta| \leqslant M} |y|^{\mu + |\alpha|}\ |\partial_x^\beta \partial_y^\alpha K(x, y)| \tag{0.1}$$

where the supremum is taken over $x \in \pi_L(C_j)$ and $y \in \imath_j(P, C'_P)$.

If $\mu = n = \dim C_P$ we consider also the seminorms

$$\|K\|_M^{\imath,j} = \sup \sum_{|\alpha| \leqslant M} \left| \int_{\varepsilon \leqslant |y| \leqslant 1} \partial_x^\alpha K(x, y)\, dy \right| \tag{0.2}$$

with the supremum taken over $0 < \varepsilon \leqslant 1$ and $x \in \pi_L(C_j)$ and $y \in \imath_j(P, C'_P)$.

We fix a C^∞ density dv on X and a C^∞ density $d\sigma$ on C. If $d\sigma_P$ is the density induced on each fiber $\{P\} \times C_P$ of $\pi_L \colon C \to X$ we consider

$$Rf(P) = \int_{C_P} K(P, Q)\, f(Q)\, d\sigma_P(Q), \qquad \text{where} \quad f \in C_0^\infty(X). \tag{0.3}$$

In this paper we give a proof of the following statements:

THEOREM 1. *Let* $\dim X = n + d$, $d = \operatorname{codim} C$ *and suppose that the rank of* $d\pi_L$ *and of* $d\pi_R$ *is at least* $n + 2d + k$ *with* $k \geqslant 1$. *Suppose* $\mu < n$. *Then for every* $r \in \mathbb{R}$, R *in* (0.3) *extends into a bounded operator*:

(1) $R\colon L_r^2(X) \to L_{r+(n-\mu)/2}^2(X)$ *if* $\mu > n - k$

(2) $R\colon L_r^2(X) \to L_{r-\varepsilon+(k/2)}^2(X)$ *for every* $\varepsilon > 0$ *if* $\mu = n - k$

(3) $R\colon L_r^2(X) \to L_{r+(k/2)}^2(X)$ *if* $\mu < n - k$.

and the norms of these operators depend, for any fixed r, *on finitely many of the seminorms in* (0.1) *corresponding to* μ.

THEOREM 2. *For every* $r \in \mathbb{R}$ *and for every* p *with* $1 < p < \infty$, R *in* (0.3) *extends into a bounded operator* $R\colon L_r^p(X) \to L_r^p(X)$, *and the norm of this operator can be bounded in terms of finitely many of the seminorms* (0.1) *corresponding to* $\mu = n = \dim C_P$ *and of finitely many of the seminorms* (0.2).

A much stronger result than Theorem 2 is proved in [1].

THEOREM 3. *Let* $n - k < \mu < n$. *Then for every* $r \in \mathbb{R}$ *and* p *with* $|\frac{1}{2} - 1/p| < \mu - n + k/2k$, R *extends into a bounded operator* $R\colon L_r^p(X) \to L_{r+(n-\mu)/2}^p(X)$ *and the norms of this operator can be bounded by means of finitely many of the seminorms in* (0.1) *corresponding to* μ.

We will consider only cases where $K(P, Q)$ is a smooth function. However, taking limits, the results extend to singular (when $\mu = n$) and fractional (when $\mu < n$) Radon transforms (see p. 106 in [7] and p. 223 in [2] for definitions). Let's discuss the sharpness of these results. Here, when we say that a result is sharp, we mean that we know of a specific operator which meets the hypothesis of one of the theorems and does not satisfy better estimates than the ones proved in that theorem. Theorem 1 cannot be improved and in particular in the case $\mu = n - k$ the statement would be false if we took $\varepsilon = 0$, all this will be discussed in Sect. 1 after the proof of Theorem 1. The sharpness of Theorem 2 can be seen from the sharpness of Theorem 1. For p in the open interval indicated in Theorem 3, the result of Theorem 3 is sharp; the author does not know if the statement remains true for the endpoints of that interval.

Singular Radon transforms where used in the study of the $\bar{\partial}$ Neumann problem, see [7]. Fractional Radon transforms (with the manifold C singular along the diagonal of $X \times X$) were applied in problems of integral geometry (see [2] for example).

The operators (0.3) were introduced in [6] and Theorems 1 and 2 were proved in [7], [8], [2] and [11] in the case when N^*C is a canonical graph (but a more general result than our Theorem 2 was proved in [1]). The proofs in [2] and [11] are very elegant but they reduce into a general fact like the L^2 boundedness of Fourier integral operators with canonical relation $(N^*C)'$ and order 0. Since we drop the condition that $(N^*C)'$ is a canonical graph, we remain closer to the arguments in [7] (and [10] where are considered Radon transforms in the case $\mu = 0$ and with weaker hypothesis on N^*C) rather than to those in [2] and [11]. We are able however to exploit more efficiently than in [7] the oscillations in certain oscillatory integrals using also the arguments on p. 172-175 in [10], thanks to a suggestion of C. D. Sogge, using a sort of Littlewood-Paley decomposition (see (1.15) below) of the operator (1.14) below. Once Theorem 1 is proved, Theorem 2 can be obtained by the discussion on p. 138-144 in [7]. Theorem 3 follows using complex interpolation by stronger versions of Theorems 1 and 2 (which are implicit in the proofs).

1. Proof of Theorem 1

We follow the same line of proof of Theorem A in [7]. Let $C_1 \subset C_2$ be small neighbourhoods of the diagonal in C and let $\chi(P, Q)$ be a C^∞ function which is 1 in C_1 and 0 outside C_2. Let's write $Rf(P) = R_1 f(P) + R_2 f(P)$ with

$$R_2 f(P) = \int (1 - \chi(P, Q))\, K(P, Q)\, f(Q)\, d\sigma_P(Q). \tag{1.1}$$

Let's consider the operator (1.1). Splitting, if necessary, this last operator in a finite sum of other operators by means of a partition of unity in supp K and focusing our attention on any of these new operators, we can suppose that the Schwartz kernel is of the form

$$\delta(\Phi(P, Q))(1-\chi(P, Q))\, K(P, Q) \qquad \text{where} \quad C=\{(P, Q): \Phi(P, Q)=0\},$$

where $\Phi: X\times X\to \mathbb{R}^d$ has maximal rank and $\delta(t)$ is the Dirac measure supported in $0\in\mathbb{R}^k$. The Schwartz kernel is just equal to

$$[1-\chi(P, Q)]\, K(P, Q)\frac{1}{(2\pi)^{d/2}}\int e^{i\langle\lambda, \Phi(P, Q)\rangle}1(\lambda)\, d\lambda,$$

with $1(\lambda)$ the constant function equal to 1. But $[1-\chi(P, Q)]\, K(P, Q)\, 1(\lambda)$ is a symbol of degree 0 (we will indicate with $S^0(X\times X, R^d)$ the set of these symbols) and as consequence the corresponding operator is a Fourier integral operator in $I^m(X, X, (N^*C)')$ with $0=m+2(n+d)/4-d/2=m+n/2$ and therefore $m=-n/2$. As a consequence (1.1) is a finite sum of Fourier integral operators of order $-n/2$ with canonical relation $(N^*C)'$. Choose now

$$\sigma=\frac{n-\mu}{2} \quad \text{or} \quad \frac{k}{2}-\varepsilon \quad (\text{with } \varepsilon>0) \quad \text{or} \quad \frac{k}{2} \tag{1.2}$$

according if μ is larger, equal or smaller than $n-k$. Consider then $(1-\Delta)^{\sigma/2}\, R_2$ which belongs to $I^{(-n/2)+\sigma}(X, X, (N^*C)')$. With our choices we have

$$\frac{-n}{2}+\sigma\leqslant\frac{2(k+d)-2(n-d)}{4}=\frac{-(n-k)}{2}. \tag{1.3}$$

Since (1.3) means

$$-\frac{n}{2}+\sigma\leqslant((rk\, d\pi_L-\dim X)+(rk\, d\pi_R-\dim X)-2\dim X)/4$$

where $\dim X=n+d$ and $rk\, d\pi_L$ and $rk\, d\pi_R$ are not smaller than $n+2d+k$ we can apply Theorem 4.32 in [3] concluding that for every $r\in\mathbb{R}$,

$$(1-\Delta)^{\sigma/2}\, R_2: L^2_r(X)\to L^2_r(X)$$

is bounded, the norm depending on finitely many of the derivatives of $[1-\chi(P, Q)]\, K(P, Q)$.

The main point of the proof of Theorem 1 lies in the study of R_1. Using a partition of unity, if necessary, we can reduce to the case where $X=\mathbb{R}^{n+d}$

and supp$[K(P, Q)\,\chi(P, Q)]$ is a very small neighbourhood of the origin in $X \times X$. With essentially the same argument as in the proof of the Corollary on p. 111 in [7], we can assume that a coordinate system $\mathbb{R}^{n+d} = \mathbb{R}^d \times \mathbb{R}^n$ is given such that

$$C = \{(t, x, s, y) \in (\mathbb{R}^d \times \mathbb{R}^n) \times (\mathbb{R}^d \times \mathbb{R}^n) \qquad \text{with} \qquad s = t + S(t, x, y)\}$$

where $S(t, x, y)\colon \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ is a C^∞ function with

$$S(t, x, x) = 0 \qquad \text{and} \qquad S(t, 0, y) = 0. \tag{1.4}$$

For completeness we give a sketch of the argument. We consider an embedding of the form $\gamma\colon \mathbb{R}^d \to \mathbb{R}^{n+d}$ defined in a small neighbourhood of $0 \in \mathbb{R}^d$, with $\gamma(0) = 0$, with γ transversal to $C_{\gamma(t)}$ and choose for each $C_{\gamma(t)}$ a coordinate system $P \in C_{\gamma(t)} \to x \in \mathbb{R}^n$ sending $\gamma(t)$ in the origin and varying smoothly with t. Then the map $\mathbb{R}^{n+d} \supseteq V \ni P \to (t, x) \in \mathbb{R}^{n+d}$ is a new coordinate system and in the new coordinates $C_{(t,0)} = \{(s, y) \in \mathbb{R}^d \times \mathbb{R}^n$ with $s = t\}$. Considering the equations defining C in $(\mathbb{R}^d \times \mathbb{R}^n) \times (\mathbb{R}^d \times \mathbb{R}^n)$ and applying the implicit function theorem we conclude that we can express C as the zero locus of $s = t + S(t, x, y)$ with $S(t, x, y)$ satisfying the properties stated above. The condition about the rank of $d\pi_L$ and $d\pi_R$ (i.e. that is at least $n + 2d + k$) is transposed in a condition on the rank of the matrix (4.1.6) p. 167 [3], that is

$$\text{if} \quad J = \begin{pmatrix} 0 & d'_x S & 1_d + d'_t S \\ d_y S & d^2_{xy}\langle \lambda, S\rangle & d^2_{ty}\langle \lambda, S\rangle \\ 1_d & 0 & 0 \end{pmatrix} \qquad \text{then} \quad \operatorname{rk} J \geqslant 2d + k$$

(where here say $d_y S$ is a $n \times d$ matrix, say $d'_y S$ is a $d \times n$ matrix and 1_d is the identity matrix of rank d) and in our case in the condition

$$\operatorname{rank}\left[\sum_{j=1}^{d} \lambda_j S_{j,\,xy}\right] \geqslant k \qquad \text{for every} \quad \lambda = (\lambda_1, ..., \lambda_d) \neq 0. \tag{1.5}$$

which follows trivially from the previous one for points where $x = 0$ using (1.4) and it remains true near these points because the rank of J is a lower-semicontinous function and does not change if we multiply λ by a positive constant. Our operator can now be written as

$$R_1 f(x, t) = \int_{\mathbb{R}^d} d\lambda \int_{\mathbb{R}^{n+d}} ds\, dy\, e^{i\langle \lambda,\, t + S(t, x, y) - s\rangle}\, K(t, x, x - y)\, f(s, y) \tag{1.6}$$

with $K(t, x, z)$ a C^∞ function with compact support satifying inequalities

$$|\partial^\alpha_{(t,x)}\, \partial^\beta_z\, K(t, x, z)| \leqslant c_{\alpha,\beta}\, |z|^{-\mu - |\beta|} \tag{1.7}$$

where the $c_{\alpha\beta}$ and the values of the seminorms (0.1) for $\chi(P, Q)\, K(P, Q)$ are essentially the same. Then Theorem 1 is a consequence of the following:

PROPOSITION 1.1. *Consider an operator,which we will call R instead of R_1, given by the RHS of* (1.6), *with $K(t, x, z)$ satisfying* (1.7) *and $S(t, x, y)$ satisfying* (1.4), (1.5). *Then R, initially defined for $f \in C_0^\infty(\mathbb{R}^{n+d})$, extends into a bounded operator for every real number r as follows*

(1) *if* $\mu > n - kR$: $L_r^2(\mathbb{R}^{n+d}) \to L^2_{r+(n-\mu)/2}(\mathbb{R}^{n+d})$

(2) *if* $\mu = n - kR$: $L_r^2(\mathbb{R}^{n+d}) \to L^2_{r+k/2-\varepsilon}(\mathbb{R}^{n+d})$ *for any* $\varepsilon > 0$

(3) *if* $\mu < n - kR$: $L_r^2(\mathbb{R}^{n+d}) \to L^2_{r+k/2}(\mathbb{R}^{n+d})$

For any fixed r, the norm of any of these operators depends on finitely many $c_{\alpha\beta}$'s in (1.7).

Proof. Let's begin with some notion that can be found in [2]. Let

$$a(t, x, s, y, \lambda, \xi) = 1(\lambda)\, \chi(s) \int e^{-i\langle \xi, z\rangle} K(t, x, z)\, dz \tag{1.8}$$

where $\chi(s) \in C_0^\infty(\mathbb{R}^d)$ and is equal to 1 in a sufficiently large neighbourhood of the origin. It is standard that

$$|\partial^\alpha_{(t, x, s, y)}\, \partial^\beta_\lambda\, \partial^\gamma_\xi\, a(t, x, s, y, \lambda, \xi)| \leqslant c_{\alpha\beta\gamma}(1 + |\lambda| + |\xi|)^{r - |\beta|}\, (1 + |\xi|)^{t - |\gamma|} \tag{1.9}$$

with $r = 0$ and $t = -n + \mu$ and where each $c_{\alpha\beta\gamma}$ can be bounded in terms of finitely many $c_{\alpha\beta}$'s in (1.7). In general, when a function $a(t, x, s, y, \lambda, \xi)$ verifies with all its derivatives (1.9) for a fixed pair of real numbers r and t, then $a(t, x, s, y, \lambda, \xi)$ is said to belong to $S^{r,t}(\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}, \mathbb{R}^d, \mathbb{R}^n)$. This last space can be provided with seminorms defined using the constants $c_{\alpha\beta\gamma}$ in (1.9).

The kernel K_R of R can be represented as an oscillatory integral

$$K_R(t, x, s, y) = \frac{1}{(2\pi)^{(d+n)/2}} \int e^{[\langle \lambda,\, t + S(t, x, y) - s\rangle + \langle x - y, \xi\rangle]} \times a(t, x, s, y, \lambda, \xi)\, d\lambda\, d\xi \tag{1.10}$$

where $a(t, x, s, y, \lambda, \xi)$ is given by (1.8). We say $K_R \in I^{r,t}_{\mathrm{comp}}(\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}, C, \Delta)$, where $r = 0$ and $t = -n + \mu$ and where $\Delta \subseteq \mathbb{R}^{n+d} \times \mathbb{R}^{n+d}$ is the diagonal. In general an element of $I^{r,t}(\mathbb{R}^m \times \mathbb{R}^m, C, \Delta)$ here $m = n + d$, is a distribution in $\mathscr{D}'(\mathbb{R}^m \times \mathbb{R}^m)$ which can be written as a locally finite sum of oscillatory integrals as in (1.10). Here we will not distinguish between spaces of operators and corresponding spaces of kernels.

LEMMA 1.2. *Let* $R \in I^{r,t}(\mathbb{R}^m \times \mathbb{R}^m, C, \Delta)$. *Then, for any pseudodifferential operator* P, *the kernels of the compositions* PR *and* RP *belong to* $I^{r+\text{order } P,\, t}(\mathbb{R}^m \times \mathbb{R}^m, C, \Delta)$.

Proof. For this proof we write $x \in \mathbb{R}^m$ instead of $(t, x) \in \mathbb{R}^{n+d}$ and y instead of (s, y). Also C is defined by the system of equations $\Phi(x, y) = 0$. Let now P be a standard pseudodifferential operator acting on $C_0^\infty(\mathbb{R}^m)$ and let $Q: C_0^\infty(\mathbb{R}^m \times \mathbb{R}^m) \to C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ be of the form $Qf(x, y) = Pf(\cdot, y)(x)$. Then Q extends in a continous linear operator $Q: \mathscr{E}'(\mathbb{R}^m \times \mathbb{R}^m) \to \mathscr{D}'(\mathbb{R}^m \times \mathbb{R}^m)$. Given now $K \in I^{r,t}_{\text{comp}}(\mathbb{R}^m \times \mathbb{R}^m, C, \Delta)$ we want to show $QK \in I^{r+\text{order } P,\, t}(\mathbb{R}^m \times \mathbb{R}^m, C, \Delta)$. We need to consider only case $K = I(a)$, where

$$I(a)(x, y) = \int e^{i[\langle \xi_1, \Phi(x, y)\rangle + \langle \xi', x' - y'\rangle]} a(x, y, \xi)\, d\xi$$

where $\xi = (\xi_1, \xi') \in \mathbb{R}^d \times \mathbb{R}^n$ and similarly $x = (x_1, x')$ and $y = (y_1, y')$, $a(x, y, \xi) = 0$ if (x, y) does not belong to a fixed compact set and

$$|\partial^\alpha_{(x, y)}\, \partial^\beta_{\xi_1}\, \partial^\gamma_{\xi'}\, a(x, y, \xi)| \leqslant c_{\alpha\beta\gamma}(1 + |\xi|)^{r - |\beta|}\, (1 + |\xi'|)^{t - |\gamma|}. \tag{A}$$

We suppose at first that $a(x, y, \xi)$ has compact support. Then

$$QK(x, y) = \int e^{i[\langle \xi_1, \Phi(x, y)\rangle + \langle \xi', x' - y'\rangle]} A(x, y, \xi)\, d\xi$$

with

$$\begin{aligned} A(x, y, \xi) &= e^{-i[\langle \xi_1, \Phi(x, y)\rangle + \langle \xi', x' - y'\rangle]} P\{e^{i[\langle \xi_1, \Phi(\cdot, y)\rangle + \langle \cdot' - y', \xi'\rangle]} a(\cdot, y, \xi)\} \\ &= \int e^{i[\langle \eta, x - z\rangle + \langle \xi_1, \Phi(x, z) - \Phi(x, y)\rangle + \langle \xi', z' - x'\rangle]} \\ &\qquad \times p(x, \eta)(a(z, y, \xi)\, d\eta\, dz. \end{aligned}$$

For $\chi \in C_0^\infty(\mathbb{R})$ an appropriate function equal to 1 in a neighbourhood of 1 and with support not containing 0, we reduce to consider

$$\begin{aligned} B(a)(x, y, \xi) = |\xi|^m \int & e^{i|\xi|[\langle \eta, x - z\rangle + \langle \xi_1/|\xi|, \Phi(z, y) - \Phi(x, y)\rangle + \langle \xi'/|\xi|, z' - x'\rangle]} \\ & \times \chi(\eta)\, p(x, |\xi|\, \eta)\, a(z, y, \xi)\, d\eta\, dz \end{aligned}$$

because it is easy to see that the other term is $O(|\xi|^{-N})$ for every N. We have

$$\begin{aligned} &|\partial^\delta_{(\eta, z)}\, \partial^\alpha_{(x, y)}\, \partial^\beta_{\xi_1}\, \partial^\gamma_{\xi'}\, \chi(\eta)\, p(x, |\xi|\, \eta)\, a(z, y, \xi)| \\ &\qquad \leqslant c_{\alpha\beta\gamma\delta}(1 + |\xi|)^{r + \text{order } P - |\beta|}\, (1 + |\xi'|)^{t - |\gamma|} \end{aligned} \tag{B}$$

where each of the constants in (B) is bounded in terms of finitely many of the constants in (A). Finally

$$|\partial^{\alpha}_{(x,\,y)}\,\partial^{\beta}_{\xi_1}\,\partial^{\gamma}_{\xi'}\,B(a)(x, y, \xi)| \leqslant c_{\alpha\beta\gamma}(1+|\xi|)^{r+\text{order }P-|\beta|}\,(1+|\xi'|)^{t-|\gamma|} \tag{C}$$

using the stationary phase theorem (see for example p. 41 in [9]): notice that the Hessian of the phase has determinant equal in absolute value to 1 and each of the constants in (C) depends on finitely many of the constants in (B). The map $S^{r,\,t}_{\text{comp}} \to \mathscr{E}'$ given by $a \to I(a)$ is continous and the map $S^{r,\,t}_{\text{comp}} \to S^{r+\text{order }P,\,t}$ given by $a \to B(a)$ is also continous. If now $a = \lim a_j$ and $QI(a_j) = IB(a_j)$, then taking the limit we have $QI(a) = IB(a)$. This gives the proof for PR. The proof for RP is similar. ▮

It is possible to split $(1-\Delta)^{s/2} = P_1 + P_2$, with P_1 a properly supported pseudodifferential operator and P_2 a smoothing one. Thanks to Lemma 1.2 it is easy to conclude that $P_2 R$ and RP_2 are smoothing pseudodifferential operators.

We turn now to the main part of the proof of Proposition 1.1 which is a consequence of the following:

Proposition 1.3. *Let R be an operator with kernel of the same type as* (1.10) *but with*

$$a(t, x, s, y, \lambda, \xi) \in S^{\sigma,\,-n+\mu}(\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}, \mathbb{R}^d, \mathbb{R}^n)$$

and σ as in (1.2). *Suppose $a(t, x, s, y, \lambda, \xi) = 0$ if (t, x, s, y) does not belong to a fixed compact set. Then $R\colon L^2(\mathbb{R}^{n+d}) \to L^2(\mathbb{R}^{n+d})$ is bounded and its norm depends on finitely many of the constants in* (1.9).

Proof. It is not restrictive to assume here $a(t, x, s, y, \lambda, \xi) = 0$ for $|\lambda| + |\xi| \leqslant 1$. Now we let $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 0$ if $t \leqslant 1$ and $\chi(t) = 1$ for $t > 2$. Write

$$a(t, x, s, y, \lambda, \xi) = a(t, x, s, y, \lambda, \xi)_1 + a(t, x, s, y, \lambda, \xi)_2$$

with

$$\begin{aligned} a(t, x, s, y, \lambda, \xi)_1 &= a(t, x, s, y, \lambda, \xi)\left[1 - \chi\left(\frac{|\xi|}{|\lambda|}\right)\right] \\ a(t, x, s, y, \lambda, \xi)_2 &= a(t, x, s, y, \lambda, \xi)\,\chi\left(\frac{|\xi|}{|\lambda|}\right) \end{aligned} \tag{1.1}$$

and consider correspondengly $R = R_1 + R_2$. It is easy to see that $a(t, x, s, y, \lambda, \xi)_2$ is a standard compound symbol of order $\sigma - (n-\mu)(\leqslant 0)$. Therefore R_2 is a standard pseudodifferential operator of nonpositive order and this concludes the discussion for this operator. For $a(t, x, s, y, \lambda, \xi)_1$ of (1.11) we have inequalities

$$|\partial^{\alpha}_{(t, x, s, y)} \partial^{\beta}_{\lambda} \partial^{\gamma}_{\xi} a(t, x, s, y, \lambda, \xi)_1| \leqslant c_{\alpha\beta\gamma}(1+|\lambda|)^{\sigma-|\beta|} (1+|\xi|)^{-n+\mu-|\gamma|} \tag{1.12}$$

Let now $K(t, x, s, y, \lambda, z) = \int e^{i\langle z, \xi\rangle} a(t, x, s, y, \lambda, \xi)_1 \, d\xi$. Then it is standard that inequalities of the following form hold

$$|\partial^{\alpha}_{(t, x, s, y)} \partial^{\beta}_{\lambda} \partial^{\gamma}_{z} K(t, x, s, y, \lambda, z)| \leqslant c_{\alpha\beta\gamma}(1+|\lambda|)^{\sigma-|\beta|} |z|^{-\mu-|\gamma|}. \tag{1.13}$$

Dropping once more the index 1 and denoting R_1 by R, we consider now

$$Rf(x, t) = \int e^{i\langle\lambda, t-s+S(t, x, y)\rangle} K(t, x, s, \lambda, x-y) \, f(s, y) \, d\lambda \, ds \, dy \tag{1.14}$$

where $K(t, x, s, y, \lambda, z)$ satisfies (1.13) and is equal to 0 if (t, x, s, y, z) does not belong to a preassigned compact set. In the discussion of this operator we will apply the ideas on p. 172–175 in [10]. We consider now a partition of unity in $\mathbb{R}^d$, $\beta_0(|\lambda|) + \sum_{k=1}^{\infty} \beta_k(|\lambda|) = 1$ with $\beta_k(|\xi|) = \beta(2^{-k} |\xi|)$ for $k \geqslant 1$ with supp $\beta \subseteq \{\lambda : \frac{1}{2} \leqslant |\lambda| \leqslant 2\}$. Let's define

$$\begin{aligned} R_k f(t, x) = \int_{\mathbb{R}^d} d\lambda \, \beta_k(|\lambda|) \int_{\mathbb{R}^{n+d}} ds \, dy \, e^{\langle\lambda, t+S(t, x, y)-s\rangle} \\ \times K(t, x, s, y, \lambda, x-y) \, f(s, y). \end{aligned} \tag{1.15}$$

Then we have:

Lemma 1.4. *There exists a constant c depending only on finitely many of the constants in* (1.9) *such that*:

(1) $\|R_k f\| \leqslant c\|f\|$ *for any* k (*by* $\|\cdot\|$ *we mean always the* L^2 *norm*)

(2) $\|R_k R_l^* f\| \leqslant c 2^{-\max(l, k)} \|f\|$ *if* $|k-l| > 3$

(3) $\|R_k^* R_l f\| \leqslant c 2^{-\max(l, k)} \|f\|$ *if* $|k-l| > 3$

Once Lemma 1.4 is proved, Proposition 1.3 follows from the Cotlar–Stein almost orthogonality theorem.

Proof. Let's prove first the second and the third claims which are easier.

Proof of 2. The kernel of the operator is

$$H(t, x, s, y) = \int d\lambda\, d\tau \int_{(v, z) \in \mathbb{R}^d \times \mathbb{R}^n} dv\, dz\, e^{i[\langle \lambda, t - v + S(t, x, z)\rangle - \langle \tau, s - v + S(s, y, z)\rangle]}$$
$$\times \{K(t, x, v, z, \lambda, x - z)\, \overline{K(s, y, v, z, \tau, y - z)}\}\, \beta_k(|\lambda|)\, \beta_l(|\tau|)$$

We perform the dv integration and we obtain the following inequalities, which are valid for any integer N and which follow from (1.13)

$$|[K(t, x, ..., z, \lambda, x - z)\, \overline{K(s, y, ..., z, \tau, y - z)}]\widehat{\ }\, (\lambda - \tau)|$$
$$\leqslant c_N (1 + |\lambda|)^\sigma\, (1 + |\tau|)^\sigma\, (1 + |\lambda - \tau|)^{-N}\, |x - z|^{-\mu}\, |y - z|^{-\mu}. \quad (1.16)$$

Taking absolute values we obtain

$$|H(t, x, s, y)| \leqslant c_N \iint d\lambda\, d\tau (1 + |\lambda|)^\sigma\, (1 + |\tau|)^\sigma\, (1 + |\lambda - \tau|)^{-N}$$
$$\times \beta_k(|\lambda|)\, \beta_l(|\tau|) \times \int_{|z| \leqslant c} dz\, |x - z|^{-\mu}\, |y - z|^{-\mu}. \quad (1.17)$$

For $|k - l| > 3$ we have $(1 + |\lambda - \tau|) \approx \max(|\lambda|, |\tau|)$ and so the first factor is bounded by $2^{-\max(l, k)}$, taking N large enough. The second factor is bounded by a multiple of $|x - y|^{-\mu}$. Therefore $|H(t, x, s, y)| \leqslant c_1 2^{-\max(l, k)}\, |x - y|^{-\mu}$ where the LHS has fixed compact support. The Young inequality implies then $\| R_k R_l^* \| \leqslant c 2^{-\max(l, k)}$. ∎

Proof of 3. The operator has kernel

$$H(t, x, s, y) = \int d\lambda\, d\tau \int_{(v, z) \in \mathbb{R}^d \times \mathbb{R}^n} dv\, dz \beta_k(|\lambda|)\, \beta_l(|\tau| \cdots)\, e^{-i\langle \lambda, v - t + S(v, z, x)\rangle}$$
$$\times e^{i\langle \lambda, v - s + S(v, z, y)\rangle}\, \overline{K(v, z, t, x, \lambda, z - x)}\, K(v, z, s, y, \tau, z - y).$$

Let

$$k(\tau, \lambda, t, x, s, y, z) = \int dv\, e^{[\langle \tau - \lambda, v\rangle + \langle \tau, S(v, z, y)\rangle - \langle \lambda, S(v, z, x)\rangle]}$$
$$\times \overline{K(v, z, t, x, \lambda, z - x)}\, K(v, z, s, y, \tau, z - y).$$

Claim. *For every positive integer N,*

$$|k(\tau, \lambda, t, x, s, y, z)|$$
$$\leqslant c_N (1 + |\lambda|)^\sigma\, (1 + |\tau|)^\sigma\, (1 + |\lambda - \tau|)^{-N}\, |x - z|^{-\mu}\, |y - z|^{-\mu}$$

Notice that the Claim implies an analogue of (1.16) and from there the proof can proceed as the previous one.

Proof of the claim. Let

$$\Delta = \left\langle \frac{\tau-\lambda}{|\tau-\lambda|}, \partial_v \right\rangle (\langle \tau-\lambda, v\rangle + \langle \tau, S(v, z, y)\rangle - \langle \lambda, S(v, z, x)\rangle)$$
$$= |\tau-\lambda| + S_v(v, z, x)(\tau-\lambda) + O(|x-y|\tau).$$

Now (1.4), the fact that we can think $|x-y|$ to be very small and that either $|\tau-\lambda| \approx |\tau|$ or $|\tau-\lambda| \gg |\tau|$, imply, that $|\Delta| \geqslant c\,|\tau-\lambda|$.

Let now $D = (1/i\Delta)\langle (\tau-\lambda)/|\tau-\lambda|, \partial_v\rangle$, and $L = D^*$. Then

$$|k(\tau, \lambda, t, x, s, y, z)| \leqslant \int dv\, |L^N\{\overline{K(v, z, t, x, \lambda, z-x)}\, K(v, z, s, y, \tau, z-y)\}|.$$

The Claim now follows from the inequalities (1.13). ∎

Proof of 1. We will discuss three distinct statements.

LEMMA 1.5. *For any real number τ greater than* 1 *let*

$$Rf(t, x) = \int_{\mathbb{R}^d} d\lambda\, \beta\left(\frac{|\lambda|}{\tau}\right) \int_{\mathbb{R}^{n+d}} ds\, dy\, e^{i\langle \lambda,\, t+S(t, x, y)-s\rangle}$$
$$\times K(t, x, s, y, \lambda, x-y)\, f(s, y) \tag{1.18}$$

where $K(\cdot)$ satisfies (1.13) *with $\sigma = n-\mu/2$ and is equal to* 0 *if (t, x, s, y) isn't in a preassigned neighbourhood of the origin*; supp $\beta \subseteq [1/2, 2]$ *and $\sum_{k \in \mathbb{Z}} \beta(2^{-k}t) = 1$ for $t > 0$, $S(\cdot)$ satisfies* (1.4) *and* (1.5) *and $\mu > n-k$. Then $\|R\| \leqslant c$, where the constant does not depend on τ but depends on finitely many of the constants in* (1.13).

LEMMA 1.6. *Same statement as in Lemma* 1.5 *but with $\sigma = k/2$ and $0 < \mu < n-k$.*

LEMMA 1.7. *Same statement as in Lemmas* 1.5 *and* 1.7 *but with $\sigma = k/2 - \varepsilon$ and $\mu = n-k$, where ε is any fixed strictly positive number.*

Lemma 1.5 is the most difficult, Lemma 1.6 is standard and together they imply Lemma 1.7 by means of an interpolation. Let's consider the proof of these lemmas in the order.

Proof of Lemma 1.5. As in [7] and in [10] let's integrate in ds. We obtain

$$\int_{\mathbb{R}^{n+d}} d\lambda\, dy\, e^{i\langle \lambda,\, t+S(t,\, x,\, y)\rangle} \int d\varrho \hat{K}(t, x, \varrho, y, \lambda, x-y)\, \tilde{f}(\lambda-\varrho, y)\, \beta\left(\frac{|\lambda|}{\tau}\right)$$

where $\tilde{f}(\varrho, y) = \int e^{-i\langle \varrho, s\rangle} f(s, y)\, ds$ and

$$\hat{K}(t, x, \varrho, y, \lambda, x-y) = \int e^{-i\langle \varrho, s\rangle} K(t, x, s, y, \lambda, x-y)\, ds. \tag{1.19}$$

From (1.13) it follows

$$\begin{aligned} &|\partial^{\alpha}_{(t,\, x,\, \varrho,\, y)}\, \partial^{\beta}_{\lambda}\, \partial^{\gamma}_{z}\, \hat{K}(t, x, \varrho, y, \lambda, z)| \\ &\qquad \leqslant c_{\alpha\beta\gamma M}(1+|\varrho|)^{-M}\, (1+|\lambda|)^{\sigma-|\beta|}\, |z|^{-\mu-|\gamma|}. \end{aligned} \tag{1.20}$$

We reduce to the following

LEMMA 1.8. *The operators*

$$\tilde{T}_{\varrho} g(t, x) = \int_{\mathbb{R}^{n+d}} e^{i\langle \lambda,\, t+S(t,\, x,\, y)\rangle} \beta\left(\frac{|\lambda|}{\tau}\right) \hat{K}(t, x, \varrho, y, \lambda, x-y)\, \tilde{g}(\lambda, y)\, d\lambda\, dy \tag{1.21}$$

have norm

$$\|\tilde{T}_{\varrho}\| \leqslant c_M(1+|\varrho|)^{-M},$$

with each c_M depending on finitely many of the constants in (1.20).

It is easy to see that Lemma 1.8 implies Lemma 1.5.

Proof of Lemma 1.8. We proceed as in the proof of Theorem 3 in [7]. Using a function $\phi \in C_0^{\infty}(\mathbb{R}^n)$ equal to 1 in a neighbourhood of 0, we write $\tilde{T}_{\varrho} = (\tilde{T}_{\varrho} - T_{\varrho}) + T_{\varrho}$ where

$$\begin{aligned} T_{\varrho}\, g(t, x) = \int_{\mathbb{R}^{n+d}} & e^{i\langle \lambda,\, t+S(t,\, x,\, y)\rangle} \beta\left(\frac{|\lambda|}{\tau}\right) \hat{K}(t, x, \varrho, y, \lambda, x-y) \\ & \times (1-\phi(|x-y|^2\, |\lambda|))\, \tilde{g}(\lambda, y)\, d\lambda\, dy. \end{aligned} \tag{1.22}$$

We write $(\tilde{T}_{\varrho} - T_{\varrho})\, g(t, x) = \int_{\mathbb{R}^d} d\lambda\, e^{i\langle \lambda, t\rangle} \beta(|\lambda|/\tau)\, a(t, \lambda, \varrho)\, \hat{g}(\lambda)$ where we are interpreting g as $g\colon \mathbb{R}^d \to L^2(\mathbb{R}^n)$ with $\hat{g}$ its Fourier transform and where $a(t, \lambda, \varrho)$ is the operator defined by

$$a(t, \lambda, \varrho)\, h(x) = \int_{\mathbb{R}^n} e^{i\langle \lambda,\, t + S(t, x, y)\rangle} \hat{K}(t, x, \varrho, y, \lambda, x-y) \times \phi(|x-y|^2\, |\lambda|)\, h(y)\, dy.$$

Now we have:

LEMMA 1.19. *For any fixed $h(x) \in C_0^\infty(\mathbb{R}^n)$, $a(t, x, \lambda, \varrho)\, h(x)$ is C^∞ in the t, λ variables and*

$$\|\partial_t^\alpha \partial_\lambda^\beta a(t, x, \lambda, \varrho)\, h\| \leqslant c_{\alpha\beta M}(1+|\varrho|)^{-M} (1+|\lambda|)^{|\alpha|/2 - |\beta|/2} \|h\|,$$

each of the constants depending on finitely anally of the constants in (1.20).

We skip the proof, which is an easier version of the proof of Lemma 1 p. 132 in [7]. As a consequence $\tilde{T}_\varrho - T_\varrho$ is a pseudodifferential operator of order 0 and type $(\frac{1}{2}, \frac{1}{2})$ and the desired estimates $\|\tilde{T}_\varrho - T_\varrho\| \leqslant c_M(1+|\varrho|)^{-M}$ follow. Let's turn to T_ϱ given by (1.22). The case $k = n$ of Proposition 1.3 can be proved in a rather straightforward way using an argument similar to that in Sect. 2 in [2] (this has been done in [11]). We will consider therefore only the case $k < n$ (but there is an essentially similar proof for $k = n$). For some of the details we refer to Chap. 2 in [6] which inspired most of the following discussion.

Let us pick $\lambda \in \mathbb{R}^d$. We reduce to the case where $(z', z'') \in \mathbb{R}^d \times \mathbb{R}^n$ with $|\det\langle \lambda, S_{x'y'}(t, x, y)\rangle| > c_0$ where $c_0 > 0$. By a continuity argument we can suppose that c_0 does not depend on λ. Let's consider also a partition of unity in S^{d-1}, $\sum \alpha_j(\lambda) = 1$ and the let's write $T_\varrho = \sum T_j$, where T_j is defined by (1.22) with $\beta(|\lambda|)$ replaced by $\alpha_j(\lambda)\, \beta(|\lambda|)$. Generically let's indicate this last function by $\beta(\lambda)$, which therefore is such that $\beta(\lambda) \neq 0$ implies that $\frac{1}{2} \leqslant |\lambda| \leqslant 2$ and that λ belongs to some thin cone. In what follows we discuss the T_j's, dropping the index j to simplify the notation. We consider the operator

$$T_{x''y''} g(t, x') = \tau^{d/2} \int e^{i\tau\langle \lambda,\, s + S(t, x', x'', y', y'')\rangle} \beta(\lambda)[1 - \phi(|x-y|^2 \tau)] \times \hat{K}(t, x', x'', \varrho, y', y'', \tau\lambda, x'-y', x''-y'')\, \tilde{g}(\lambda, y')\, dy'\, d\lambda \tag{1.23}$$

LEMMA 1.10.

$$\|T_{x''y''}\| \leqslant c_M(1+|\varrho|)^{-M} \tau^{(n-k)/2} (1 + \tau\, |x'' - y''|^2)^{\mu/2}$$

with each c_M depending on finitely many constants in (1.20).

Lemma 1.10 and Young's inequality imply Lemma 1.8 and therefore also Lemma 1.5.

Proof of Lemma 1.10. The kernel of $T_{x''y''}T^*_{x''y''}$ is

$$H(t, x', s, y') = \tau^d \int d\lambda\, dz'\, e^{\tau i\langle \lambda,\, t - s + S(t, x', x'', z', y'') - S(s, y', x'', z', y'')\rangle} \mathscr{A}$$

where

$$\begin{aligned} \mathscr{A} = {} & \beta^2(\lambda)[1 - \phi((|x' - z'|^2 + |x'' - y''|^2)\,\tau)] \\ & \times [1 - \phi((|y' - z'|^2 + |x'' - y''|^2)\,\tau)] \\ & \times \hat{K}(t, x', x'', \varrho, z', y'', \tau\lambda, x' - z', x'' - y'') \\ & \times \overline{\hat{K}(s, y', x'', \varrho, z', y'', \tau\lambda, y' - z', x'' - y'')}. \end{aligned}$$

If now supp $\beta(\lambda)$ is inside a sufficiently thin cone, as we may suppose, there exists a unitary $a \in \mathbb{R}^{d+k}$ of the form $a = ((t - s)/|t - s|, B(x' - y')/|x' - y'|)$ where B is say the inverse of $\langle \lambda_0, S_{x'y'}(0, 0, 0)\rangle$ such that, if

$$\Delta = \langle a, \partial_{\lambda, z'}\rangle \{\langle \lambda, t - s + S(t, x', x'', z', y'') - S(s, y', x'', z', y'')\rangle\},$$

we have $|\Delta| \geqslant c(|t - s| + |x' - y'|)$ for some $c > 0$. Let $D = 1/(i\tau\,\Delta)\langle a, \partial_{\lambda, z'}\rangle$ and let $L = D^*$. Then

$$H(t, x', s, y') = \tau^d \int e^{i\tau\langle\cdot\rangle} L^N\{\mathscr{A}\}\, d\lambda\, dz' \tag{1.24}$$

Claim.

$$\begin{aligned} |\partial_\lambda^\alpha[L_N\{\mathscr{A}\}]| \leqslant {} & c_{\alpha NM}(1 + |\varrho|)^{-M}\,\tau^{2\sigma - N}(|t - s| + |x' - y'|)^{-N} \\ & \times \sum_{l + k = N} (|x' - z'|^2 + |x'' - y''|^2)^{(-\mu - l)/2} \\ & \times (|y' - z'|^2 + |x'' - y''|^2)^{(-\mu - k)/2} \end{aligned} \tag{1.25}$$

each $c_{\alpha NM}$ depending on finitely many of the constants in (1.20).

Proof. The proof is elementary and follows from (1.20) and from the inequalities

$$\left|\partial_\lambda^\delta\, \partial_{z'}^\varepsilon \left[\frac{\prod_j \partial_{z'}^{\beta_j}\, \partial_\lambda^{\gamma_j}\, \Delta}{\Delta^{N+M}}\right]\right| \leqslant c_{\delta\varepsilon N}\, \frac{1}{|\Delta|^N} \qquad \text{if} \quad \beta(\lambda) \neq 0$$

where $M \leqslant \sum(|\beta_j| + |\gamma_j|) \leqslant N$. These last inequalities are essentially a consequence of the case $\delta = 0$ and $\varepsilon = 0$. But for $\delta = 0$ and $\varepsilon = 0$ they follow because

$$c_{\beta\gamma}\,|\Delta| \geqslant \tilde{c}_{\beta\gamma}(|t-s| + |x'-y'|) \geqslant |\partial_{z'}^{\beta}\,\partial_{\lambda}^{\gamma}\,\Delta|. \quad \blacksquare$$

We integrate in (1.24) in $d\lambda$, basically calculating a Fourier transform, and we obtain

$$\tau^d \int dz' \{ [\widehat{L^N \mathscr{A}}](\tau(t-s+S(t, x', x'', z', y'') - S(s, y', x'', z', y''))) \} \tag{1.26}$$

Taking the absolute value in (1.26) and using (1.25), we can write

$$\begin{aligned} |H(t, x', s, y')| \leqslant c_{NMK}(1+|\varrho|)^{-M}\, \tau^{d+2\sigma-N}(|t-s| + |x'-y'|)^{-N} \\ \times \int_{\tilde{c} \geqslant |x'-z'|^2 + |x''-y''|^2 \geqslant c/\tau,\ \tilde{c} \geqslant |y'-z'|^2 + |x''-y''|^2 \geqslant c/\tau} dz' \\ \times (1+\tau\,|t-s+S(t, x', x'', z', y'') - S(s, y', x'', z', y'')|)^{-K} \\ \times \sum_{l+k=N} (|x'-z'|^2 + |x''-y''|^2)^{(-\mu-l)/2} \\ \times (|y'-z'|^2 + |x''-y''|^2)^{(-\mu-k)/2} \end{aligned} \tag{1.27}$$

We turn now to the following:

Claim.

$$\sup_{(s,\, y')} \int |H(t, x', s, y')|\, dt\, dx' \leqslant c_M(1+|\varrho|)^{-M}\, \tau^{n-k}(1+\tau|x''-y''|^2)^{-\mu} \tag{1.28}$$

$$\sup_{(t,\, y')} \int |H(t, x', s, y')|\, ds\, dy' \leqslant c_M(1+|\varrho|)^{-M}\, \tau^{n-k}(1+\tau\,|x''-y''|^2)^{-\mu} \tag{1.29}$$

Notice that by Young inequality the Claim implies Lemma 1.8.

Proof of Claim. We discuss here only (1.28) since basically the same proof yields (1.29). Let's fix (s, y') and use (1.27) for $N = k-1$ and $N = k+1$. For $a = \tau^{-1/2}(1+\tau\,|x''-y''|^2)^{-1/2}$

$$\int |H(t, x', s, y')|\, dt\, dx'$$

$$\leqslant c_M(1+|\varrho|)^{-M} \int_{|x'| \leqslant a} dx'\, \tau^{-\mu+n-k+1}\, |x'|^{-k+1}$$

$$\times \int_{\tilde{c} \geqslant |z'|^2 + |x''-y''|^2 \geqslant c/\tau} dz'\, (|z'|^2 + |x''-y''|^2)^{(-2\mu-k+1)/2}\, \mathscr{B}$$

$$+ c_M(1+|\varrho|)^{-M} \int_{|x'| \geqslant a} dx'\, \tau^{-\mu+n-k-1}\, |x'|^{-k-1}$$

$$\times \int_{\tilde{c} \geqslant |z'|^2 + |x''-y''|^2 \geqslant c/\tau} dz'\, (|z'|^2 + |x''-y''|^2)^{(-2\mu-k-1)/2}\, \mathscr{B} + \mathscr{A}$$

with

$$\mathscr{B} = \int dt\, \tau^d (1+\tau|t-s+S(t, x', x'', \tilde{z}, y'') - S(s, y', x'', \tilde{z}, y'')|)^{-K}$$

where $\tilde{z} = z' + x$ and $\mathscr{A}$ is a sum of two analogous terms but where in the corresponding $\mathscr{B}$ we have $\tilde{z} = z' + y$. We now have for K sufficiently large:

Subclaim. *$\mathscr{B} \leqslant c$ with c a constant not depending on any of the parameters in the integral defining $\mathscr{B}$.*

With the Subclaim we reduce to the situation considered on p. 40 in [6].

Proof of the Subclaim. Let us write $\mathscr{B}$ as

$$\int_{\mathbb{R}^d} dt (1+\tau\, |t-s+A(t, t-s)+B|)^{-K} \tag{1.30}$$

with

$$B = S(s, x', x'', \tilde{z}, y'') - S(s, y', x'', \tilde{z}, y'')$$

and

$$A(t, t-s) = \int_0^1 S_t(s+r(t-s), x', x'', \tilde{z}, y'')(t-s)\, dr.$$

Then

$$\frac{\partial}{\partial u} A(t, u) = \int_0^1 S_{tt}(s + ru, x, \tilde{z}, y'')\, ur\, dr + \int_0^1 S_t(s + ru, x, \tilde{z}, y'')\, dr.$$

Because of (1.4) and the fact that $|u| = |t-s|$ is bounded, we can assume $|A(t, s)| \ll 1$ and $|(\partial/\partial u)\, A(t, u)| \ll 1$. Performing in (1.30) the change of variable $v = t - s + A(t, t-s) - B$ we obtain that (1.30) is essentially bounded by $\int_{\mathbb{R}^d} \tau^d (1 + \tau |v|)^{-K}\, dv$ and this gives the Subclaim.

Proof of the Claim: *Continuation.* We are reduced to

$$\begin{aligned}
&\int |H(t, x', s, y')|\, dt\, dx' \\
&\quad \leqslant c_M (1 + |\varrho|)^{-M} \int_{|x'| \leqslant a} dx'\, \tau^{-\mu + n - k + 1}\, |x'|^{-k+1} \\
&\quad \times \int_{\tilde{c} \geqslant |z'|^2 + |x'' - y''|^2 \geqslant c/\tau} dz'\, (|z'|^2 + |x'' - y''|^2)^{(-2\mu - k + 1)/2} \\
&\quad + c_M (1 + |\varrho|)^{-M} \int_{|x'| \geqslant a} dx'\, \tau^{-\mu + n - k - 1}\, |x'|^{-k-1} \\
&\quad \times \int_{\tilde{c} \geqslant |z'|^2 + |x'' - y''|^2 \geqslant c/\tau} dz'\, (|z'|^2 + |x'' - y''|^2)^{(-2\mu - k - 1)/2} \qquad (1.31)
\end{aligned}$$

with $a = \tau^{-1/2} (1 + |x'' - y''|^2)^{-1/2}$. What follows is taken from [6]. We have

$$\begin{aligned}
&\int_{\tilde{c} \leqslant |z'|^2 + |x'' - y''|^2 \leqslant c/\tau} dz'\, (|z'|^2 + |x'' - y''|^2)^{(-2\mu - k + 1)/2} \\
&\quad \leqslant C \tau^{(2\mu - 1)/2} (1 + |x'' - y''|^2)^{(-2\mu + 1)/2} \int_{\mathbb{R}^k} dz'\, (1 + |z'|)^{(-2\mu - k + 1)/2}.
\end{aligned}$$

The last integral is convergent because $2\mu + k - 1 > k$. Similarly

$$\begin{aligned}
&\int_{\tilde{c} \geqslant |z'|^2 + |x'' - y''|^2 \geqslant c/\tau} dz'\, (|z'|^2 + |x'' - y''|^2)^{(-2\mu - k - 1)/2} \\
&\quad \leqslant C \tau^{(2\mu - 1)/2} (1 + |x'' - y''|^2)^{(-2\mu - 1)/2} \int dz'\, (1 + |z'|)^{(-2\mu - k - 1)/2}.
\end{aligned}$$

Therefore the RHS of (1.31) is bounded by

$$\int |H(t, x', s, y')|\, dt\, dx'$$

$$\leqslant c_M (1+|\varrho|)^{-M}\, \tau^{n-k+(1/2)} (1+|x''-y''|^2)^{(-2\mu+1)/2} \int_{|x'|\leqslant a} |x'|^{-k+1}\, dx'$$

$$+ c_M (1+|\varrho|)^{-M}\, \tau^{n-k-(1/2)} (1+|x''-y''|^2)^{(-2\mu-1)/2} \int_{|x'|\leqslant a} |x'|^{-k-1}\, dx'$$

$$\leqslant c_M (1+|\varrho|)^{-M}\, \tau^{n-k} (1+|x''-y''|^2)^{\mu}. \quad \blacksquare$$

Proof of Lemma 1.6. As in Lemma 1.5 we are reduced to consider operators (1.21). As above we are reduced to

$$R_{x''y''} g(t, x')$$

$$= \tau^{d/2} \int_{\mathbb{R}^{n+d}} e^{\tau i\langle \lambda,\, t+S(t, x', x'', y', y'')\rangle}$$

$$\times \beta(\lambda)\, \hat{K}(t, x', x'', \varrho, y', y'', \lambda, , x'-y', x''-y'')\, \tilde{g}(\lambda, y')\, d\lambda\, dy'.$$

Now, for τ sufficiently large, this operator has norm bounded by an expression of the form $(1+|\varrho|)^{-M}\, |x''-y''|^{-\mu}$ [7, Theorem 3]. Lemma 1.6 then follows using Minkovsky and Young inequalities. $\blacksquare$

Proof of Lemma 1.7. It follows by complex interpolation considering an analytic family of operators of the form

$$R_\gamma g(t, x) = \exp(\gamma^2) \int_{\mathbb{R}^{n+d}} e^{i\langle \lambda,\, t+S(t, x, y)\rangle}\, \beta\left(\frac{|\lambda|}{\tau}\right) (1-\phi(|x-y|^2\, |\lambda|))$$

$$\times \hat{K}(t, x, \varrho, y, \lambda, x-y)\, |x-y|^{\gamma}\, \tilde{g}(\lambda, y)\, d\lambda\, dy. \quad \blacksquare$$

We will give now examples of specific operators which don't satisfy better estimates than those proved in Theorem 1. For the first two claims (1) and (2) let's consider operators $Rf(t, x) = \int f(t-A(y), x-y)\, \eta(y)\, |y|^{-\mu}\, dy$ where $t \in \mathbb{R}^d$, x and y belong to $\mathbb{R}^n$, $\eta \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the origin and $A: \mathbb{R}^n \to \mathbb{R}^d$ is a vector valued quadratic form such that for any nonzero $\lambda \in \mathbb{R}^d$ $\partial^2/\partial y^i \partial y^j \langle \lambda, A(y)\rangle \geqslant k$ with the equality valid for some values of λ. If $I(\lambda, \xi) = \int e^{i[\langle \lambda, A(y)\rangle + \langle \xi, y\rangle]} \times \eta(y)\, |y|^{-\mu}\, dy$ we ask if the absolute value of $I(\lambda, \xi)$ is bounded by $|(\lambda, \xi)|^{-\sigma-a}$ with either $a>0$ and $\sigma = n-\mu/2$ if $\mu > n-k$ or $a=\varepsilon$ and $\sigma = ((n-\mu)/2) - \varepsilon$ if $\mu = n-k$. Since $I(\lambda, \xi)$ is a smooth function this would imply that $I(\lambda, 0)$ were bounded by $|\lambda|^{-\sigma-a}$ when restricted in a halfline where $\partial^2/\partial y^i\,\partial y^j\langle\lambda, A(y)\rangle = k$. For the case $\mu = n-k$ it is shown in [5] that the above bound cannot hold. Also the bound for the case $\mu > n-k$ cannot hold as can be shown arguing as follows. Consider the operator $T_\tau f(x) = \int e^{i\tau B(x-y)}\eta(y)\,|x-y|^{-\mu} f(y)\,dy$ where τ is a large real positive number and B is a quadratic form whose corresponding symmetric matrix has rank k. Then if the L^2 norm of T_τ were bounded by $\tau^{-\sigma-a}$ one could show that $Tf(x) = \int e^{iB(x-y)}\,|x-y|^{-\mu} f(y)\,dy$ is the constant operator equal to 0, and easily one can understand that this is not true. The third claim (3) in Theorem 1 is also sharp as can be seen considering again $I(\lambda, \xi)$ defined as above but with $\mu = 0$.

2. Proof of Theorem 2

As above we consider $Rf(P) = R_1 f(P) + R_2 f(P)$ with R_2 defined as in (1.1). We discuss R_2 first. As operator (1.1) above, R_2 is a Fourier Integral Operator of order $-n/2$ (recall $\dim X = n+d$). Since (1.3) holds when $\sigma = 0$ for any $k \geqslant 1$ we conclude by Theorem 4.32 in [3] that for any $r \in \mathbb{R}$ our R_2 extends into a continous operator from $L^2_r(X)$ to itself. The fact that $R_2\colon L^p_0(X) \to L^p_0(X)$ for any p with $1 < p < \infty$ is discussed on p. 144 in [7] and we omit the proof here. Finally the L^p_r boundedness for generic r follows by interpolation.

The main part of the proof deals with R_1. Essentially we are reduced to the case when the kernel K_R of R has support in an arbitrarily small neighbourhood of the diagonal in $X \times X$. We reduce to prove the following analogue of Proposition 1.3.

Proposition 2.1. *Let R be an operator with kernel*

$$K_R(t, x, s, y) = \int e^{i[\langle\lambda,\, t+S(t,x,y)-s\rangle + \langle x-y,\, \xi\rangle]} a(t, x, s, y, \lambda, \xi)\,d\lambda\,dy\,d\xi$$

with

$$a(t, x, s, y, \lambda, \xi) \in S^{0,0}(\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}, \mathbb{R}^d, \mathbb{R}^n)$$

where $a(t, x, s, y, \lambda, \xi) = 0$ if (t, x, s, y) does not belong to a fixed compact set in $\mathbb{R}^{n+d} \times \mathbb{R}^{n+d}$. Then $R\colon L^p(\mathbb{R}^{n+d}) \to L^p(\mathbb{R}^{n+d})$ is a bounded operator for any $1 < p < \infty$.

Proof. We proceed exactly as in Proposition 1.3 until we reduce to consider the following operator

$$R_1 f(t, x) = \int e^{i\langle\lambda,\, t-s+S(t,x,y)\rangle}\, K(t, x, s, y, \lambda, x-y)\, f(s, y)\,d\lambda\,dy \quad (2.1)$$

where $K(t, x, s, y, \lambda, x-y)=0$ for $|\lambda|<1$,

$$|\partial^{\alpha}_{(t, x, s, y)} \partial^{\beta}_{\lambda} \partial^{\gamma}_{z} K(t, x, s, y, \lambda, z)| \leqslant c_{\alpha\beta\gamma}(1+|\lambda|)^{-|\beta|} |z|^{-n-|\gamma|},$$

where

$$\left| \int_{\varepsilon \leqslant |z| \leqslant c} \partial^{\alpha}_{z} K(t, x, s, y, \lambda, z) \, dz \right| \leqslant c_{\alpha}$$

for some fixed c and any $\varepsilon > 0$ and where $K(t, x, s, y, \lambda, z)$ is 0 if (t, x, s, y, z) does not belong to a preassigned compact set. We fix $\phi \in C_0^{\infty}(\mathbb{R}^n)$ with $\phi = 1$ in a neighbourhood of the origin and define an analytic family of operators

$$T_{\gamma} = T^1 + T^2_{\gamma} \tag{2.2}$$

$$T^1 f(t) = \int_{\mathbb{R}^d} e^{i\langle \lambda, t\rangle} a^1(t, \lambda, \varrho) \hat{f}(\lambda) \, d\lambda \tag{2.3}$$

$$T^2_{\gamma} f(t) = \int_{\mathbb{R}^d} e^{i\langle \lambda, t\rangle} a^2_{\gamma}(t, \lambda, \varrho) \hat{f}(\lambda) \, d\lambda \tag{2.4}$$

where we interpret $f: \mathbb{R}^d \to L^2(\mathbb{R}^n)$ (here say $f \in C_0^{\infty}(\mathbb{R}^{n+d})$) and $a^1(t, \lambda, \varrho)$ resp. $a^2_{\gamma}(t, \lambda, \varrho)$ are operators having kernel

$$\phi(|x-y|^2 |\lambda|) \, e^{i\lambda S(t, x, y)} \hat{K}(t, x, \varrho, y, \lambda, x-y),$$

resp.

$$[1-\phi(|x-y|^2 |\lambda|)](|\lambda| \, |x-y|^2)^{-\gamma} \, e^{i\lambda S(t, x, y)} \hat{K}(t, x, \varrho, y, \lambda, x-y) \tag{2.5}$$

where

$$\hat{K}(t, x, \varrho, y, \lambda, , z) = \int e^{-i\langle \varrho, s\rangle} K(t, x, s, y, \lambda, z) \, ds$$

with

$$|\partial^{\alpha}_{(t, x, \varrho, y)} \partial^{\beta}_{\lambda} \partial^{\gamma}_{z} \hat{K}(t, x, \varrho, \lambda, , z)| \leqslant c_{\alpha\beta\gamma M}(1+|\varrho|)^{-M} (1+|\lambda|)^{-|\beta|} |z|^{-n-|\gamma|} \tag{2.6}$$

and

$$\int_{\varepsilon \leqslant |z| \leqslant c} \partial^{\alpha}_{z} \hat{K}(t, x, \varrho, y, \lambda, , z) \, dz| \leqslant c_{\alpha M}(1+|\varrho|)^{-M} \tag{2.7}$$

for a fixed c and for any $\varepsilon > 0$ (see pp. 131–132 in [7]). The case $p=2$ in Proposition 2.1 follows, by the Minkovsky inequality, from the following:

LEMMA 2.2. *For $\gamma = 0$ we have*

$$\|T_\gamma\| \leqslant c_M (1+|\varrho|)^{-M}.$$

Proof. We begin discussing the operator T^1 given by (2.3). The following holds:

LEMMA 2.3. *$\partial_t^\alpha \partial_\lambda^\beta a^1(t, \lambda, \varrho)$ extend into bounded operators in $L^2(\mathbb{R}^n)$ and*

$$\|\partial_t^\alpha \partial_\lambda^\beta a^1(t, \lambda, \varrho)\| \leqslant c_{\alpha\beta M}(1+|\varrho|)^{-M} (1+|\lambda|)^{(|\alpha|/2)-(|\beta|/2)}$$

The proof is exactly that of Lemma 1 on p. 132 in [7]. As a consequence

$$\|T^1\| \leqslant c_M(1+|\varrho|)^{-M}.$$

We turn now to the operator T_γ^2 in (2.4). Here the discussion will be a much easier version of the corresponding one on pp. 134-139 in [7]. First of all we have an elementary partial analogue of Lemma 2 on p. 134 in [7].

LEMMA 2.4. *Consider the operators $a_\gamma^2(t, \lambda, \varrho)$ whose kernels are given by* (2.5). *For any positive integer C these exists a $C_1 > 0$ such that if* $\operatorname{Re} \gamma > C_1$ *then for every pair of multiindexes α, β with $|\alpha| + |\beta| < C$ we have*

$$|\partial_t^\alpha \partial_\lambda^\beta a_\gamma^2(t, \lambda, \varrho)| \leqslant c_M(1+|\varrho|)^{-M} (1+|\lambda|)^{(|\alpha|/2)-(|\beta|/2)} \tag{2.8}$$

where the c_M are polynomial functions of γ.

Proof. We have

$$a_\gamma^2(t, \gamma, \varrho)\, f(x) = \int e^{i\lambda S(t,x,y)}[1-\phi(|x-y|^2\,|\lambda|)]$$
$$\times (|\lambda|\,|x-y|^2)^{-\gamma}\, \hat{K}(t, x, \varrho, y, \lambda, x-y)\, f(y)\, dy. \tag{2.9}$$

To estimate the norm of this operator we simply invoke Young's inequality

$$\|a_\gamma^2(t, \lambda, \varrho)\| \leqslant \tilde{c}_M(1+|\varrho|)^{-M}\, |\lambda|^{\operatorname{Re}\gamma} \int_{\tilde{c}\,|\lambda|^{-1/2} \leqslant |x| \leqslant c} |x|^{-n-2\operatorname{Re}\gamma}\, dx$$
$$\leqslant c_M(1+|\varrho|)^{-M}$$

where for the first inequality we use (2.6). Now $\partial_t^{\alpha} \partial_\lambda^{\beta} a_\gamma^2(t, \gamma, \varrho) f(x)$ is a sum of terms of the form

$$\int \partial_\lambda^{\beta_1} \partial_t^{\alpha_1} e^{i\lambda S(t, x, y)} \partial_\lambda^{\beta_2}[1-\phi(|x-y|^2 |\lambda|)] \partial_\lambda^{\beta_3} |\lambda|^{-\operatorname{Re}\gamma} |x-y|^{-2 \operatorname{Re}\gamma}$$

$$\times \partial_\lambda^{\beta_4} \partial_t^{\alpha_2} \hat{K}(t, x, \varrho, y, \lambda, x-y) f(y)\, dy$$

which by (2.6) are essentially of the form

$$\int e^{i\lambda S(t, x, y)} \{[1-\phi(\cdot)]^{(|\beta_2|)} (|x-y|^2 |\lambda|)\} \Phi_{|\alpha_1| - \operatorname{Re}\gamma - |\beta_3| - |\beta_4|}(\lambda)$$

$$\times |x-y|^{-2 \operatorname{Re}\gamma + |\alpha_1| + |\beta_1| + 2|\beta_2| - n} k(t, x, \varrho, y, x-y) f(y)\, dy$$

where $\Phi_m(\lambda)$ is an homogeneous function of degree m and $k(t, x, \varrho, y, z)$ can be thought as a homogeneous function of degree 0 in z. We use (2.6) and Young's inequality and we obtain upper bounds of the form

$$c_M (1+|\varrho|)^{-M} |\lambda|^{|\alpha_1| - \operatorname{Re}\gamma - |\beta_3| - |\beta_4| - n} \int_{\tilde{c}|\lambda|^{-1/2} \leqslant |x| \leqslant c} |x|^{l+n-1}\, d|x|$$

where $l = -2 \operatorname{Re}\gamma + |\alpha_1| + |\beta_1| + 2|\beta_2| - n$. Now, if, $-2 \operatorname{Re}\gamma + |\alpha| + 2|\beta| < 0$, these are essentially bounded by

$$(1+|\varrho|)^{-M} |\lambda|^{|\alpha_1| - \operatorname{Re}\gamma - |\beta_3| - |\beta_4|} |\lambda|^{\operatorname{Re}\gamma - |\alpha_1|/2 - |\beta_1|/2 - |\beta_2|}$$

$$\leqslant (1+|\varrho|)^{-M} |\lambda|^{|\alpha_1|/2 - |\beta|_1/2 - |\beta_2| - |\beta_3| - |\beta_4|}$$

and therefore all of these are bounded by $(1+|\varrho|)^{-M} (1+|\lambda|)^{|\alpha|/2 - |\beta|/2}$. ∎

Thanks to Lemma 2.4 we conclude, using the theory of pseudodifferential operators, that if $a > 0$ is sufficiently large, for $\operatorname{Re}\gamma = a$, the operators (2.4) satisfy

$$\|T_\gamma^2\| \leqslant c_M (1+|\varrho|)^{-M}$$

where the c_M are polinomials in γ.

LEMMA 2.5. *Fix b with $n-k < 2b+n < n$ and suppose* $\operatorname{Re}\gamma = b$. *Then*

$$\|T_\gamma^2\| \leqslant c_M (1+|\varrho|)^{-M},$$

each c_M depending polynomially on $\operatorname{Im}\gamma$.

Lemma 2.5 follows from the proof of Theorem 1. Using Lemmas 2.4 and 2.5 we obtain by interpolation that

$$\|\exp(\gamma^2)\, T_\gamma\| \leqslant c_M(1+|\varrho|)^{-M} \tag{2.10}$$

for $b \leqslant \operatorname{Re}\gamma \leqslant a$ with $a>0$ sufficiently large and b such that $n-k<2b+n<n$. This in particular gives Lemma 2.2.

We turn now to the generic p with $1<p<\infty$ in Proposition 2.1. Here the argument goes as on pp. 140–144 in [7]. We consider again the operators (2.2) which we write as $T_\gamma f(P)=\int K(P,Q)\, f(Q)\, dQ$. Then, for every fixed $P=(t,x)$ and $Q=(s,y)$, we define new coordinates for the Q writing $Q=[\sigma,z]$ with

$$\sigma = s-t-S(t,x,y)$$

$$z=y-x.$$

We write $\exp(\gamma^2)\, K_\gamma(P,Q)=M(P,\sigma,z)$ and we have:

Lemma 2.6. *Let* $\alpha=\operatorname{Re}\gamma>0$. *Then if* $|\sigma|\leqslant|z|^2$ *we have*:

(1) $|M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |z|^{-n-2\alpha}\, |\sigma|^{-d+\alpha}$

(2) $|\partial_z M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |z|^{-n-1-2\alpha}\, |\sigma|^{-d+\alpha}$

(3) $|\partial_\sigma M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |z|^{-n-2\alpha}\, |\sigma|^{-d-1+\alpha}$

For $|\sigma|\geqslant|z|^2$ *we have*:

(1) $|M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |\sigma|^{-n/2-d}$

(2) $|\partial_z M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |\sigma|^{-n/2-d-1/2}$

(3) $|\partial_\sigma M(P,\sigma,z)| \leqslant c_M(1+|\varrho|)^{-M}\, |\sigma|^{-n/2-d-1}$

The proof is basically the same of Lemma 3 on p. 140 in [7].

We define now a quasidistance as follows: if $P=(t,x)$ and $Q=(s,y)$ we write $d(P,Q)<\delta$ (with $\delta\leqslant 1$) if $|x-y|<\delta$ and $|t-s+S(t,x,y)|<\delta^2$. Then:

Lemma 2.1. *Again let* $\operatorname{Re}\gamma=\alpha>0$. *Then*

$$\int |\exp(\gamma^2)|\, |K_\gamma(P,Q_1)-K_\gamma(P,Q_2)|\, dP \leqslant c_M(1+|\varrho|)^{-M} \tag{2.11}$$

where the integral is taken over the region $d(P,Q_1)\geqslant c d(Q_1,Q_2)$ *and* $c\geqslant 1$, *where* c *and* c_M *don't depend on* $\operatorname{Im}\gamma$.

The proof is basically the same of Lemma 4 on p. 141 in [7].

As a consequence of (2.10) and (2.11) we conclude that if $\operatorname{Re}\gamma=\alpha>0$ is fixed, then

$$\|\exp(\gamma^2)\, T_\gamma f\|_p \leqslant c_{pM}(1+|\varrho|)^{-M}\, \|f\|_p \qquad \text{for} \quad 1<p\leqslant 2 \qquad (2.11)$$

for every $f\in L^p(\mathbb{R}^{n+d})$ and for every M. Finally Proposition 2.1 follows by complex interpolation using Lemma 2.5 and (2.11) and by a duality argument.

3. Proof of Theorem 3

We consider first what happens when the Schwartz kernel $K_R(P, Q)$ has support disjoint from the diagonal. First of all, using the arguments at the beginning of the proof of Proposition 1.1, we observe that R is a Fourier Integral Operator of order $-n/2$, we apply Theorem 4.39 in [3] and we conclude that $R\colon L^2_r(\mathbb{R}^{n+d})\to L^2_{r+k/2}(\mathbb{R}^{n+d})$. The boundedness of $R\colon L^p_r(\mathbb{R}^{n+d})\to L^p_r(\mathbb{R}^{n+d})$ for any $r\in\mathbb{R}$ and for any $1<p<\infty$ was discussed at the beginning of the proof of Theorem 2. The result then follows by interpolation considering the analytic family of operators $(1-\Delta)^{z/2}\, R$.

We now consider what happens near the diagonal. Using the arguments at the beginning of the proof of Proposition 1.3 we are reduced to the following. We have to prove the L^P boundedness of an operator R with kernel given by (1.10) where

$$|\partial^\alpha_{(t,\,x,\,s,\,y)}\,\partial^\beta_\lambda\,\partial^\gamma_\xi\, a(t,x,s,y,\lambda,\xi)| \leqslant c_{\alpha\beta\gamma}(1+|\lambda|)^{((n-\mu)/2)-|\beta|}\,(1+|\xi|)^{-n+\mu-|\gamma|}$$

as in (1.9) and where $a(t, x, s, \lambda, \xi)$ has support inside $|\lambda|\geqslant c|\xi|$, $|\xi|\geqslant 1$ for any fixed $c>0$. Let's define

$$a_z(t,x,s,y,\lambda,\xi)=\exp(z^2)\, a(t,x,s,y,\lambda,\xi)(1+|\lambda|)^{-z/2}\,(1+|\xi|)^z.$$

For the corresponding family of operators, Proposition 2.1 implies

$$\|R_z\|_p\leqslant c_p \qquad \text{if} \quad \operatorname{Re} z=n-\mu \qquad \text{and} \qquad 1<p<\infty.$$

By Proposition 1.1 we have

$$\|R_z\|_2\leqslant c_{\operatorname{Re} z} \qquad \text{for} \quad n-\mu\geqslant \operatorname{Re} z>n-k-\mu.$$

The wanted result follows by interpolation.

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